Math 235 - Dr. Miller - Exam #1 - Feb. 24, 2012 - **SOLUTIONS**

**This exam is worth 100 points; save plenty of time for the proofs at the end.**

1. *[18 pts - 6 each]* Identify each sentence below as a true statement, a false statement, or not a statement, informally justifying your claim. A sentence or two will suffice.

   (a) There exists a real number $a$ for which, for every real number $b$, $|ab| < 1$.

   *True statement - The real number $a = 0$ has the property that, for every real number $b$, $|ab| = 0 < 1$.*

   (b) The numbers $p$ and $q$ are real numbers with $|pq| = 1$.

   *Not a statement - A statement must be a declarative sentence that is definitely either true or false. This sentence will be true for some substitutions of p and q but false for others, violating the definition of a statement. (It's actually a predicate.)*

   (c) For every real number $x$, there exists a real number $y$ for which $|xy| > 1$.

   *False statement - the real number 0 cannot be multiplied by any real value $y$ to make an absolute value other than 0, which is not greater than 1.*

2. *[15 pts - 5 each]* Write the hypothesis only - as a present-tense, stand-alone sentence - of each implication below.

   (a) The number $g$ can only be positive if $g + 1$ is also positive.

   *The number $g$ is positive.*

   (b) For $h - 1$ to be negative, it is necessary that $h$ be less than 1.

   $h - 1$ *is negative.*

   (c) A sufficient condition for $1 - k^2$ to be positive is that $|k|$ be less than 1.

   $|k|$ *is less than* 1.

3. *[10 pts]* Use a standard truth table to confirm that $((A \Leftrightarrow B) \wedge \sim A) \Rightarrow \sim B$ is a tautology. Clearly indicate the appropriate column in your table.

| $A$ | $B$ | $A \Leftrightarrow B$ | $\sim A$ | $(A \Leftrightarrow B) \wedge \sim A$ | $\sim B$ | $((A \Leftrightarrow B) \wedge \sim A) \Rightarrow \sim B$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

4. *[21 pts - 7 each]* Negate each statement below in as positive a form as possible.

(a) If $xy = yz$, then $y = 0$ or $x > z$. (The numbers $x$, $y$, and $z$ are real.)

*There exist real numbers $x$, $y$, and $z$ for which $xy = yx$ but $y \neq 0$ and $x \leq z$.*

(b) For every positive integer $x$, $\frac{1}{x} < 1 < x$.

*There exists a positive integer $x$ for which $\frac{1}{x} \geq 1$ or $1 \geq x$.*

(c) If there exists a real number $a$ for which $a^2 + \frac{1}{a} < 1$, then $a$ is negative.

*There exists a real number $a$ for which $a^2 + \frac{1}{a} < 1$, but $a \geq 0$.*

5. *[12 pts]* Prove rigorously (i.e., be thorough and formal): there exist two composite numbers whose sum is prime.

*Proof.*

*Consider the numbers* 8 *and* 9*. They are composite because each is an integer greater than* 1 *having a positive factor other than* 1 *or itself: for* 8, 2 *is such a factor, and for* 9*, it is* 3.

*Their sum,* 17*, is prime because it is an integer greater than* 1*, and it has no positive factors other than* 1 *and itself.*

6. *[25 pts]* Prove rigorously: for every prime number $p$, $\sqrt[p]{p}$ – that, is, the $p$th root of $p$ – is irrational. (You may use the fact that $\sqrt[p]{p}$ is positive.)

*Proof.*

*Suppose there exists a prime number $p$ for which $\sqrt[p]{p}$ is rational. We know that $\sqrt[p]{p}$ is positive.*

*Then there exist positive integers $a$ and $b$ for which $\sqrt[p]{p} = \frac{a}{b}$. Via algebra, we obtain $pb^p = a^p$.*

*Because $p$ is prime and therefore an integer greater than 1 and $b^p$ is a positive integer raised to a positive integer power, the number $pb^p$ is a positive integer greater than 1 and so can be prime factored.*

*When we do so, the left-hand side will have prime factors $p$ showing; specifically, the number of them will be a multiple of $p$ (possibly 0 if $b$ has no factors of $p$) plus 1.*

*On the right-hand side, the number of prime factors $p$ showing will be a multiple of $p$ (again, possibly 0 if $a$ has no factors of $p$).*

*It is impossible for a multiple of $p$ plus 1 to equal another multiple of $p$, for if so, algebra/arithmetic would show that 1 is the difference of two multiples of $p$ and therefore itself a multiple of $p$.*

*This means we have two different prime factorizations of our number $pb^p$, a contradiction to the Fundamental Theorem of Arithmetic (FTA).*

*Thus, there is no such prime $p$; rather, for every prime number $p$, $\sqrt[p]{p}$ is indeed irrational.*