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Math 235 - Dr. Miller - Exam #1, In-Class Portion, Spring 2023 - Friday, Feb. 17, 2023

Follow directions carefully; work in the space provided. This in-class part of the exam is worth 85 points.

There are 3 full proofs. Each is marked with **[P]**. There are also 3 launches, each marked **[La]**.

1. *[15 pts - 5 each]* Precisely negate each statement below. Don't worry about whether the statements are true or not.

   (a) $x^2 - 5x + 6 \geq 0$ only (then) if $x \leq -2$ or $x \geq 3$.

   (-1 ftt)

   $x^2 - 5x + 6 \geq 0$ and $x > -2$ and $x < 3$.

   (language variations possible)

   (-1) kept if

   (b) $y = 7$, and $x^2 = 9$ if $|x| = 3$.

   $y \neq 7$, or $|x| = 3$ but $x^2 \neq 9$.

   (-2) reversed.

   (c) There exists $n \in \mathbf{Z}$ where $nx > 1$ for all $x \in \mathbf{R}^+$.

   (-1) reversed $\mathbb{Z}$, $\mathbb{R}^+$ order.

   For all $n \in \mathbb{Z}$, $nx \leq 1$ for some $x \in \mathbb{R}^+$.

   (language variations possible, but NOT logic variations.)

2. (a) *[2 pts]* Write the logical equivalence governing proof by cases.

   (-2) other style.

   $$(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$$

   (b) *[2 pts]* Write the logical equivalence governing two-part proof (of biconditional statements).

   $$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$\frac{12}{12}$

3. *[12 pts]* **[P]** Use the formal definition of $<$ to write a rigorous direct proof of the statement below. (I'll give you the definition, for a deduction.)

Proposition: Let $p, q, x, y \in \mathbb{R}$. *If* $p < q$ *and* $x < y$, *then* $p + x < q + y$.

Pf - Let $p, q, x, y \in \mathbb{R}$.

Assume $p < q$ and $x < y$.

(NTS: $p + x < q + y$, ~~meaning~~ $p + x + \boxed{pc} = q + y$)

optional

Now $p + c = q$ for some $c \in \mathbb{R}^+$,

and $x + d = y$ for some $d \in \mathbb{R}^+$.

We add to obtain

$$p + x + c + d = q + y.$$

Since $\mathbb{R}^+$ is closed under addition, $c + d \in \mathbb{R}^+$.

(-1) $\mathbb{Z}$

Thus, $p + x < q + y$.

therefore, if $p < q$ and $x < y$, $p + x < q + y$

12/12

4. *[12 pts]* **[P]** Prove rigorously, using direct proof. (I'll give a hint, for a deduction.)

Let $x, y \in \mathbf{Z}$ have the same remainder on division by 3. If that remainder is not 0, then $3 \mid (xy - 1)$.

Pf. - Let $x, y \in \mathbb{Z}$ have the same remainder on division by 3.

Assume that remainder is not 0.

(−1) accidental loss

(NTS: $3 \mid (xy-1)$, meaning $3 \cdot \boxed{\text{int}} = xy - 1$)

Then the remainder can only be 1 or 2.

Case 1: Assume the remainder is 1.

Then $x = 3k+1$ and $y = 3l+1$ for some $k, l \in \mathbb{Z}$.

So $xy - 1 = 9kl + 3k + 3l + 1 - 1$
$= 3(3kl + k + l)$.

We have $3kl + k + l \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under mult. and addition.

Thus, $3 \mid xy - 1$.

Case 2: Assume the remainder is 2.

Then $x = 3k+2$, $y = 3l + 2$ for some $k, l \in \mathbb{Z}$.

So $xy - 1 = 9kl + 6k + 6l + 4 - 1$
$= 3(3kl + 2k + 2l + 1)$.

(−2) lost

Now $3kl + 2k + 2l + 1 \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under mult. and addition.

Thus, $3 \mid xy - 1$.

Therefore, if x, y's remainder is not 0, then $3 \mid xy - 1$.

12/12

5. *[12 pts]* **[P]** Prove by any meaningful style. (Surprise: $\geq$ algebra allowed, but for a deduction.)

Proposition: Let $t \in \mathbf{R}$. If $|t| \geq 5$, then $2t + 8 \neq 0$.

(Remember that formal $<$ definition is NEVER required for concrete numbers.)

Pf - Let $t \in \mathbb{R}$.
Assume $2t + 8 = 0$.
(NTS: $|t| < 5$)
By algebra, $t = -4$.
Since $|-4| = 4 < 5$,
we have $|t| < 5$, as desired.
Therefore, if $|t| \geq 5$, then $2t + 8 \neq 0$.

$\frac{16}{16}$

6. *[16 pts - 8 each]* Consider this Proposition: *Let $x, y \in \mathbf{Z}$. If xy is even, then x is even or y is even.*

(a) **[La]** Write the launch, up to and including one meaningful sentence BEYOND the NTS line, of a proof by contrapositive. **Do NOT complete the proof.**

pf - Let $x, y \in \mathbb{Z}$.
Assume x is odd and y is odd.
(NTS: xy is odd, meaning $xy = 2 \cdot \boxed{int} + 1$)
Then $x = 2k+1$ for some $k \in \mathbb{Z}$.

⋮

(b) **[La]** Write the launch, up to and including one meaningful sentence BEYOND the NTS line, of a proof by "or conclusion" style. **Do NOT complete the proof.**

pf - Let $x, y \in \mathbb{Z}$.
Assume xy is even and x is odd.
(NTS: y is even, meaning $y = 2 \cdot \boxed{int}$.)
Then $xy = 2n$ for some $n \in \mathbb{Z}$.
(also correct to use odd defn on x)

⋮

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7. Consider this Proposition: *Let $m \in \mathbf{R}$. and let $f(x) = \arctan x$ and $g(x) = mx - \frac{\pi}{2}$. The graphs of f and g DON'T intersect if and only if $m = 0$.*

(a) *[2 pts]* Ignoring the universal hypothesis for now, write the "if" direction in unchanged order.

The graphs of f and g DON'T intersect if $\underbrace{m=0}_{\text{hypoth.}}$.

(b) **[La]** *[8 pts]* Including the universal hypothesis, write the launch, up to and including one meaningful sentence BEYOND the NTS line, of a proof by contradiction for the "if" direction. **Do NOT complete the proof.**

(-1) if not relevant somehow

Pf – Let $m \in \mathbb{R}$ and let $f(x) = \arctan x$ and $g(x) = mx - \frac{\pi}{2}$.

(-2) bad negation

($\Leftarrow$) Assume $m = 0$ and the graphs of f and g DO intersect.

(NTS: any ✱)

(-1) kept function variables.

(various next) Then there exists a point (a, b) with $a, b \in \mathbb{R}$ that lies on both graphs at once.

⋮

8. *[4 pts]* Formally state the Fundamental Theorem of Arithmetic (FTA).

Every integer greater than 1 has exactly one prime factorization, up to the order of those factors.