Math 235 - Dr. Miller - Exam #3 **SOLUTIONS** - 04/29/14

#1-2 are proofs, and #3-5 have multiple parts. Budget your time accordingly.

1. *[25 pts]* Prove that $10 \mid 9^{2n} - 11^n$ for all natural numbers n.

   Proof. Consider the statement $10 \mid 9^{2n} - 11^n$. *When* $n = 1$, *the right-hand expression equals* 70, *which is divisible by* 10. *Assume that* $10 \mid 9^{2k} - 11^k$ *for some natural number* k. *Then there exists an integer* x *for which* $10x = 9^{2k} - 11^k$. *Now consider* $9^{2(k+1)} - 11^{k+1}$. *We have*

$$\begin{aligned} 9^{2(k+1)} - 11^{k+1} &= 81 \cdot 9^{2k} - 11^{k+1} \\ &= 81 \cdot (9^{2k} - 11^k) + 81 \cdot 11^k - 11^{k+1} \\ &= 81 \cdot 10x + 70 \cdot 11^k \qquad (\text{by } IHOP) \\ &= 10(81x - 7 \cdot 11^k). \end{aligned}$$

   Now 11^k *is an integer because* k *is a natural number, so* $81x - 7 \cdot 11^k$ *is also an integer, being the product and difference of integers. Thus,* $10 \mid 9^{2(k+1)} - 11^{k+1}$, *and by PMI,* $10 \mid 9^{2n} - 11^n$ *for all natural numbers* n.

2. In $\mathbf{R}$, define xRy if there exists $t > 0$ for which $x + t = y$.

   (a) *[5 pts]* Give examples of 3 ordered pairs that belong to R.

   Various. For example, $(1, 4)$ *belongs to* R *with* $t = 3 > 0$.

   (b) *[15 pts]* Determine whether R is transitive, and prove your claim.

   Proof. Suppose xRy *and* yRz. *Then there exist* $t > 0$ *and* $s > 0$ *with* $x + t = y$ *and* $y + s = z$. *By substitution,* $x + t + s = z$. *Because the sum of positive numbers is positive,* $s + t > 0$, *whence* xRz, *as desired, and* R *IS transitive.*

3. *[16 pts - 8 each]* Consider the relation $R = \{(2, 3),\ (3, 4),\ (4, 1)\ (4, 4)\}$ defined on $A = \{1, 2, 3, 4\}$.

   (a) R is not an equivalence relation: it's missing some necessary pairs. Give one ordered pair that's missing, and explain why it's needed. (One sentence will suffice.)

   Various. For instance $(1, 1)$, $(2, 2)$, *and* $(3, 3)$ *are missing - they would be needed to make* R *reflexive.*
   Also, $(3, 2)$, $(4, 3)$, *and* $(1, 4)$ *are needed in order to make* R *symmetric.*
   Finally, to make R *transitive, at the very least* $(2, 4)$ *must be included (because* $2R3$ *and* $3R4$*).*

   (b) Give another ordered pair that's missing for a DIFFERENT reason, and explain why it's needed.

   See above.

4. *[15 pts]* Give an example of a family of three sets that is a partition of $\{1,2,3,4,5\}$, explaining how you know. (One or two sentences is sufficient.)

   Various: for example, $\{1,2\}$, $\{3\}$, *and* $\{4,5\}$. *These are a partition because (1) none of them are empty, (2) their union is all of the given set, and (3) any two of them are disjoint.*

5. *[24 pts - 8 each]* Find $\bigcap A_i$ and $\bigcup A_i$ for each of the families below. Tell which is which.

   (a) The index set is $\mathbf{N}$, and $A_i = \left[\frac{1}{i}, 2+\frac{1}{i}\right]$.

   List a few: $A_1 = [1,3]$, $A_2 = [\frac{1}{2}, 2\frac{1}{2}]$, $A_3 = [\frac{1}{3}, 2\frac{1}{3}]$, *etc. Use a number line to help visualize.*

   $$\bigcap A_i = [1,2] \qquad\qquad \bigcup A_i = (0,3]$$

   (b) The index set is $\mathbf{Z} \setminus \{0\}$, and $A_i = \left[\frac{1}{i}, 2+\frac{1}{i}\right]$.

   This family includes all the sets above - $A_1 = [1,3]$, $A_2 = [\frac{1}{2}, 2\frac{1}{2}]$, $A_3 = [\frac{1}{3}, 2\frac{1}{3}]$, *etc. - and also those with negative subscripts. List a few of those:* $A_{-1} = [-1,1]$, $A_{-2} = [-\frac{1}{2}, 1\frac{1}{2}]$, $A_{-3} = [-\frac{1}{3}, 1\frac{2}{3}]$, *etc. Use a number line to help visualize.*

   $$\bigcap A_i = \{1\} \qquad\qquad \bigcup A_i = [-1,3]$$

   (c) The index set is $\mathbf{R}^+$ (the set of positive real numbers), and A_i is the INSIDE of the rectangle in the plane having vertices at $(0,0)$, $(0,i)$, $(i,0)$, and (i,i). You may describe your answers verbally.

   Draw some examples: A_1 *is the inside of the square of side length* 1 *"cornered" at the origin.* A_π *is the inside of the square of side length* π *in the same position.* $A_{0.01}$ *is the inside of the square whose side length is only* 0.01 *also in that position.*

   $$\bigcap A_i = \emptyset$$

   Their overlap (intersection) is the empty set, because any potential point of overlap could always be excluded from a square that was tiny enough, and we're not counting the boundaries, so the corner point $(0,0)$ *is never included.*

   $$\bigcup A_i = \textit{ Quadrant I}$$

   Their union is the entire first quadrant, because you can just make larger and larger squares to cover any potential point in the union.